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ANALYTICAL SOLUTION OF EIGENVALUE EQUATION AND CUTOFF FREQUENCIES FOR OPTICAL FIBERS WITH A NEGATIVE INDEX OF REFRACTION

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The paper presents the eigenvalue equation for a step index fiber formed by a core with a negative index of refractio ($n_{co} < 0$) and a usual dielectric cladding ($n_{cl} > 0$) and considers the influence of negative core permittivity ($\varepsilon_{co} < 0$) and permeability ($\mu_{co} < 0$) on the cutoff conditions of the transverse electrical (*TE*), transverse magnetic (*TM*) and hybrid (*EH, HE*) modes. The solutions of the dispersion equation are compared with those of the equation describing the conventional optical step index fiber. The mathematical proof is given for the hybrid mode (HE_{11}) existence with theoretical cutoff frequency equal to zero.

1. INTRODUCTION

The history of materials with negative index of refraction begins in the year 1967 when Victor Veselago published his article [1] about electromagnetic wave propagation through medium with simultaneously negative dielectric permittivity ε and magnetic permeability μ. Veselago predicted that when both ε and μ are negative the refraction index n is also negative, and that in such a medium the group and phase velocities are anti-parallel. This means that the wave vector $\vec{\kappa}$ and Poynting's vector $\vec{S}$ are anti-parallel, and we can observe a number of unusual optical effects – such as the inverted Doppler effect, Cherenkov's radiation and Snell's law. We will describe the medium as "Left-Handed Material" (LHM) [2, 3, 4] because the wave vector and vectors $\vec{E}$ and $\vec{H}$ form a left-hand triple.

The interest in the LHM media and negative refraction phenomena is confirmed by a growing number of scientific publications. The hypothetical material introduced by Veselago can nowadays be realized using artificially constructed metamaterials. The first composite medium with simultaneously negative values of effective permittivity and permeability was demonstrated by Smith *et al.* [5] in the microwave range. Trying to reach negative light beam refraction at optical wavelengths, several photonic crystal (PC) structures have been studied. For instance, in paper [6] it was reported that a planar two-dimensional PC can, as a result of negative refraction inside the crystal structure, act as a flat lens at the 3rd telecommunication window wavelength λ = 1.55 µm (see also [7–10]). Exploring the LHM flat lens theory, Pendry suggested [11, 12] that the medium *does* amplify the amplitude of evanescent waves; respectively, there is a possibility to overcome the diffraction limit.

One of the possible applications of left-handed materials is the fiber optics. The present paper theoretically studies the dispersion equation of a step index fiber with an LHM core. As far as is known, such a fiber construction has not yet been experimentally realized, however its appearance could soon be expected. That is why it is very important to find theoretical solutions of the problem now which might be compared with experimental results in the future.

2. DISPERSION EQUATION

To analyze the electromagnetic wave propagation through a fiber, we have to consider Maxwell's equations that give the relationships between the electric and magnetic fields. The fiber's cladding is formed by a conventional dielectric material and we will call it "Right-Handed Material" (RHM). Let us assume that there is a core with permittivity $-|\varepsilon_{co}|$ and permeability $-|\mu_{co}|$, and a cladding with these parameters being ε_{cl} and μ_{cl}, while $|n_{co}|>|n_{cl}|$. The radius of the core is equal to a. The RHM and LHM media are isotropic.

The fiber construction is shown in Fig. 1.

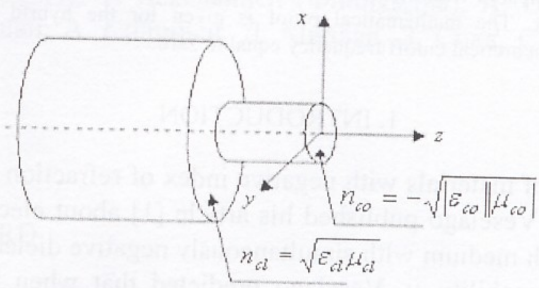

Fig. 1. Optical fiber construction

Maxwell's equations for harmonic waves ($\sim e^{j\omega t}$) take the following forms for the core:

$$\begin{cases} rot\dot{\vec{H}}_{co} = -j\omega|\varepsilon_{co}|\varepsilon_0\dot{\vec{E}}_{co} \\ rot\dot{\vec{E}}_{co} = j\omega|\mu_{co}|\mu_0\dot{\vec{H}}_{co} \end{cases} \qquad (1)$$

and for the cladding:

$$\begin{cases} rot\dot{\vec{H}}_{cl} = j\omega\varepsilon_{cl}\varepsilon_0\dot{\vec{E}}_{cl} \\ rot\dot{\vec{E}}_{cl} = -j\omega\mu_{cl}\mu_0\dot{\vec{H}}_{cl}. \end{cases} \qquad (2)$$

Besides, we will define cylindrical coordinates (ρ, φ, z) with the z-axis lying along the axis of the waveguide. If the electromagnetic waves are to propagate along the z-axis, they will obey the following functional dependences:

$$\dot{\vec{E}}(\rho,\varphi,z)=\dot{\vec{E}}(\rho,\varphi)e^{-j\beta z}$$
$$\dot{\vec{H}}(\rho,\varphi,z)=\dot{\vec{H}}(\rho,\varphi)e^{-j\beta z}, \tag{3}$$

which are harmonic in the z-coordinate. Parameter β is the z-component of the propagation vector, and ω is the wave angular frequency. The point above the electric and magnetic field strengths means the complex amplitude.

From the solutions of Eqs. (1, 2) and boundary conditions such as tangential components of electrical field inside, $E_{coZ}, E_{co\varphi}$, and outside, $E_{clZ}, E_{cl\varphi}$ of the core-cladding interface at $\rho = a$ must be the same, i.e. $E_{coZ} = E_{clZ}, E_{co\varphi} = E_{cl\varphi}$. Similarly, for the tangential components of magnetic field $H_{coZ} = H_{clZ}$, $H_{co\varphi} = H_{cl\varphi}$ [13], we can obtain the dispersion equation:

$$\left[\frac{|\mu_{co}|}{\chi_{co}}\frac{J'_m(\chi_{co}a)}{J_m(\chi_{co}a)} - j\frac{\mu_{cl}}{\chi_{cl}}\frac{H'^{(1)}_m(j\chi_{cl}a)}{H^{(1)}_m(j\chi_{cl}a)}\right]\times$$
$$\times\left[\frac{|\varepsilon_{co}|}{\chi_{co}}\frac{J'_m(\chi_{co}a)}{J_m(\chi_{co}a)} - j\frac{\varepsilon_{cl}}{\chi_{cl}}\frac{H'^{(1)}_m(j\chi_{cl}a)}{H^{(1)}_m(j\chi_{cl}a)}\right] = \left(\frac{1}{\chi_{co}^2}+\frac{1}{\chi_{cl}^2}\right)^2\frac{\beta^2 m^2}{k_0^2 a^2}, \tag{4}$$

where

$$\chi_{co}^2 = \omega^2|\varepsilon_{co}|\varepsilon_0|\mu_{co}|\mu_0 - \beta^2 = k_0^2 n_{co}^2 - \beta^2; \tag{5}$$

$$\chi_{cl}^2 = \beta^2 - \omega^2\varepsilon_{cl}\varepsilon_0\mu_{cl}\mu_0 = \beta^2 - k_0^2 n_{cl}^2; \tag{6}$$

$$k_0 = \omega\sqrt{\varepsilon_0\mu_0}. \tag{7}$$

The quantity k_0 is the vacuum wave number; $J_m(\chi_{co}a)$ is the Bessel function, and $H^{(1)}_m(j\chi_{cl}a)$ is the Hankel function of the first kind. The constant m is the index of Bessel's and Hankel's functions and this must be an integer ($m = 0,1,2\ldots$), since the fields have to be periodic in φ with a period of 2π. The prime indicates differentiation with respect to the argument $\chi_{co}a$ or $j\chi_{cl}a$ of the Bessel and Hankel functions, respectively. The coefficients χ_{co} and χ_{cl} are transverse components of the wave vector. If we change minus signs in each bracket in the left part of Eq.(4), we can obtain the conventional fiber dispersion equation (see [14], p. 296 or [15], p.10).

For large values of χ_{cl} the field is tightly concentrated inside the core and close to it [14]. With values of χ_{cl} decreasing, the field spreads further out into the space, outside the core. Finally, for $\chi_{cl} = 0$, the field detaches itself from the guide [14]. The frequency at which this happens is called the cutoff frequency. The cutoff condition is thus

$$\chi_{cl}^2 = \beta^2 - k_0^2 n_{cl}^2 = 0. \tag{8}$$

Equation (4) allows the eigenvalue solution to be obtained for the guided modes of the cladded optical fiber with a negative index of the refraction core. That is why we can use both terms: the eigenvalue equation – because Eq. (4) determines the eigenvalue β of the guided modes, and the dispersion equation – because it helps us to understand the mechanism of wave and material dispersion.

In order to simplify further consideration, we will write $\varepsilon_{co}, \mu_{co}$ instead of $|\varepsilon_{co}|$ and $|\mu_{co}|$, assuming that the core permittivity and permeability are taken by absolute values.

For different values of m we can derive different cutoff solutions. That is why we divide the dispersion equation analysis into three parts depending on the constant m values: $m>1$, $m=1$, and $m=0$.

3. CUTOFF CONDITIONS FOR $m > 1$

Equation (4) is a transcendental equation that is generally solved by numerical methods. Here it is pertinent to give partial analytical solution. One can conclude from Eqs. (5, 6) that the Bessel and Hankel functions are independent of the permittivity and permeability signs. As is written in the previous chapter, the dispersion equation for a conventional fiber differs from that of our case only by two minus signs, whereas the structure is the same. This means that we should examine how the minus signs in the left part of Eq. (4) influence the cutoff frequencies.

When $m = 2,3,4\ldots$, the solution of the eigenvalue equation right at the cutoff can be obtained by a procedure introduced by Schlesinger *et al.* [16] and demonstrated by Marcuse [14]. Following Marcuse's method, the left part of Eq. (4) can be rewritten differently:

$$\left[\frac{a}{2}\cdot\frac{\varepsilon_{co}\chi_{cl}^2}{\varepsilon_{cl}\chi_{co}}\cdot\frac{J_{m-1}(\chi_{co}a)-J_{m+1}(\chi_{co}a)}{J_m(\chi_{co}a)}-j\frac{a\chi_{cl}}{2}\frac{H_{m-1}^{(1)}(j\chi_{cl}a)-H_{m+1}^{(1)}(j\chi_{cl}a)}{H_m^{(1)}(j\chi_{cl}a)}\right]\times$$
$$\times\left[\frac{a}{2}\cdot\frac{\mu_{co}\chi_{cl}^2}{\mu_{cl}\chi_{co}}\cdot\frac{J_{m-1}(\chi_{co}a)-J_{m+1}(\chi_{co}a)}{J_m(\chi_{co}a)}-j\frac{a\chi_{cl}}{2}\frac{H_{m-1}^{(1)}(j\chi_{cl}a)-H_{m+1}^{(1)}(j\chi_{cl}a)}{H_m^{(1)}(j\chi_{cl}a)}\right], \quad (9)$$

where both parts of Eq. (4) were multiplied by $\chi_{co}^2\chi_{cl}^2a^2$ and the relationship from the cylindrical function theory [17]

$$Z'_m(z)=\frac{1}{2}\left(Z_{m-1}(z)-Z_{m+1}(z)\right) \quad (10)$$

was used. Here $Z_m(z)$ is either Bessel's or Hankel's function and z is the argument of a corresponding function.

We will introduce the abbreviations [14]:

$$\begin{cases} J^+ = \dfrac{1}{\chi_{co}a}\dfrac{J_{m+1}(\chi_{co}a)}{J_m(\chi_{co}a)} \\ J^- = \dfrac{1}{\chi_{co}a}\dfrac{J_{m-1}(\chi_{co}a)}{J_m(\chi_{co}a)} \end{cases} \quad \begin{cases} H^+ = \dfrac{1}{j\chi_{cl}a}\dfrac{H_{m+1}^{(1)}(j\chi_{cl}a)}{H_m^{(1)}(j\chi_{cl}a)} \\ H^- = \dfrac{1}{j\chi_{cl}a}\dfrac{H_{m-1}^{(1)}(j\chi_{cl}a)}{H_m^{(1)}(j\chi_{cl}a)} \end{cases} \tag{11}$$

which make it possible to rewrite dispersion equation (4) in the form:

$$\left[\frac{\varepsilon_{co}}{\varepsilon_{cl}}\left(J^- - J^+\right) + \left(H^- - H^+\right)\right] \times \left[\frac{\mu_{co}}{\mu_{cl}}\left(J^- - J^+\right) + \left(H^- - H^+\right)\right] = \\ = \left[\frac{2k_0 n_{cl}\beta m}{a^2\chi_{co}^2\chi_{cl}^2}\left(\frac{n_{co}^2}{n_{cl}^2} - 1\right)\right]^2 . \tag{12}$$

Here the right part of Eq. (4) was multiplied by $4/(a\chi_{cl})^2$.

Regrouping elements leads to the following expression:

$$\left[\left(\frac{\varepsilon_{co}}{\varepsilon_{cl}}J^- + H^-\right) - \left(\frac{\varepsilon_{co}}{\varepsilon_{cl}}J^+ + H^+\right)\right] \times \left[\left(\frac{\mu_{co}}{\mu_{cl}}J^- + H^-\right) - \left(\frac{\mu_{co}}{\mu_{cl}}J^+ + H^+\right)\right] = \\ = \left[\frac{2k_0 n_{cl}\beta m}{a^2\chi_{co}^2\chi_{cl}^2}\left(\frac{n_{co}^2}{n_{cl}^2} - 1\right)\right]^2 . \tag{13}$$

Opening the brackets we obtain:

$$\overbrace{\left(\frac{\varepsilon_{co}}{\varepsilon_{cl}}J^- + H^-\right)\left(\frac{\mu_{co}}{\mu_{cl}}J^- + H^-\right)}^{1} - \overbrace{\left(\frac{\varepsilon_{co}}{\varepsilon_{cl}}J^- + H^-\right)\left(\frac{\mu_{co}}{\mu_{cl}}J^+ + H^+\right)}^{2} - \\ - \overbrace{\left(\frac{\varepsilon_{co}}{\varepsilon_{cl}}J^+ + H^+\right)\left(\frac{\mu_{co}}{\mu_{cl}}J^- + H^-\right)}^{3} + \underbrace{\left(\frac{\varepsilon_{co}}{\varepsilon_{cl}}J^+ + H^+\right)\left(\frac{\mu_{co}}{\mu_{cl}}J^+ + H^+\right)}_{4} = \\ = \left[\frac{2k_0 n_{cl}\beta m}{a^2\chi_{co}^2\chi_{cl}^2}\left(\frac{n_{co}^2}{n_{cl}^2} - 1\right)\right]^2 . \tag{14}$$

We will number the elements in the left part with 1, 2, 3 and 4. Another relation of the cylindrical function is [17]:

$$Z_{m+1}(z) + Z_{m-1}(z) = \frac{2m}{z}Z_m(z). \tag{15}$$

Using abbreviations (11) we can obtain:

$$J^{+}+J^{-}=\frac{1}{\chi_{co}a}\cdot\frac{J_{m+1}(\chi_{co}a)+J_{m-1}(\chi_{co}a)}{J_m(\chi_{co}a)}=\frac{2m}{(\chi_{co}a)^2}\frac{J_m(\chi_{co}a)}{J_m(\chi_{co}a)}=\frac{2m}{(\chi_{co}a)^2}; \quad (16)$$

$$H^{+}+H^{-}=\frac{1}{j\chi_{cl}a}\cdot\frac{H^{(1)}_{m+1}(j\chi_{cl}a)+H^{(1)}_{m-1}(j\chi_{cl}a)}{H^{(1)}_m(j\chi_{cl}a)}=-\frac{2m}{(\chi_{cl}a)^2} \quad (17)$$

With the help of Eqs. (16) and (17) we will transform the 1st element of Eq. (14) as

$$\left(\frac{\varepsilon_{co}}{\varepsilon_{cl}}J^{-}+H^{-}\right)\left(\frac{\mu_{co}}{\mu_{cl}}J^{-}+H^{-}\right)=$$

$$=\left(-\frac{\varepsilon_{co}}{\varepsilon_{cl}}J^{+}+\frac{\varepsilon_{co}}{\varepsilon_{cl}}\frac{2m}{(\chi_{co}a)^2}-H^{+}-\frac{2m}{(\chi_{cl}a)^2}\right)\left(\frac{\mu_{co}}{\mu_{cl}}J^{-}+H^{-}\right)=$$

$$=-\underbrace{\left(\frac{\varepsilon_{co}}{\varepsilon_{cl}}J^{+}+H^{+}\right)\left(\frac{\mu_{co}}{\mu_{cl}}J^{-}+H^{-}\right)}_{5}+\underbrace{\frac{2m}{a^2}\left(\frac{\varepsilon_{co}}{\varepsilon_{cl}}\cdot\frac{1}{\chi_{co}^2}-\frac{1}{\chi_{cl}^2}\right)\left(\frac{\mu_{co}}{\mu_{cl}}J^{-}+H^{-}\right)}_{6}. \quad (18)$$

Analogously, we obtain transformation for the 4th element:

$$\left(\frac{\varepsilon_{co}}{\varepsilon_{cl}}J^{+}+H^{+}\right)\left(\frac{\mu_{co}}{\mu_{cl}}J^{+}+H^{+}\right)=$$

$$=\left(-\frac{\varepsilon_{co}}{\varepsilon_{cl}}J^{-}+\frac{\varepsilon_{co}}{\varepsilon_{cl}}\frac{2m}{(\chi_{co}a)^2}-H^{-}-\frac{2m}{(\chi_{cl}a)^2}\right)\left(\frac{\mu_{co}}{\mu_{cl}}J^{+}+H^{+}\right)=$$

$$=-\underbrace{\left(\frac{\varepsilon_{co}}{\varepsilon_{cl}}J^{-}+H^{-}\right)\left(\frac{\mu_{co}}{\mu_{cl}}J^{+}+H^{+}\right)}_{7}+\underbrace{\frac{2m}{a^2}\left(\frac{\varepsilon_{co}}{\varepsilon_{cl}}\cdot\frac{1}{\chi_{co}^2}-\frac{1}{\chi_{cl}^2}\right)\left(\frac{\mu_{co}}{\mu_{cl}}J^{+}+H^{+}\right)}_{8}. \quad (19)$$

We will number the elements in the right part of Eqs. (18, 19) with 5, 6, 7 and 8. The 5th element of Eq. (18) is equal to the 3rd element of Eq. (14), and the 7th element of Eq. (19) – to the 2nd element of Eq. (14). Using this condition, we can rewrite once more the dispersion equation as

$$-2\left(\frac{\varepsilon_{co}}{\varepsilon_{cl}}J^{-}+H^{-}\right)\left(\frac{\mu_{co}}{\mu_{cl}}J^{+}+H^{+}\right)-2\left(\frac{\varepsilon_{co}}{\varepsilon_{cl}}J^{+}+H^{+}\right)\left(\frac{\mu_{co}}{\mu_{cl}}J^{-}+H^{-}\right)+$$

$$+\frac{2m}{a^2}\left(\frac{\varepsilon_{co}}{\varepsilon_{cl}}\cdot\frac{1}{\chi_{co}^2}-\frac{1}{\chi_{cl}^2}\right)\underbrace{\left(\frac{\mu_{co}}{\mu_{cl}}J^{+}+H^{+}+\frac{\mu_{co}}{\mu_{cl}}J^{-}+H^{-}\right)}_{9}=$$

$$=\left[\frac{2k_0 n_{cl}\beta m}{a^2\chi_{co}^2\chi_{cl}^2}\left(\frac{n_{co}^2}{n_{cl}^2}-1\right)\right]^2 \quad (20)$$

The element number 9 in the left part of Eq. (20) was derived from elements 8 and 6. In this consideration we assume that the core's permittivity and permeability are taken by absolute values. Therefore we can state the following:

$$\frac{\mu_{co}}{\mu_{cl}}\left(J^{+}+J^{-}\right)+H^{+}+H^{-}=\frac{\mu_{co}}{\mu_{cl}}\frac{2m}{(\chi_{co}a)^{2}}-\frac{2m}{(\chi_{cl}a)^{2}}=\frac{2m}{a^{2}}\left(\frac{\mu_{co}}{\mu_{cl}}\frac{1}{\chi_{co}^{2}}-\frac{1}{\chi_{cl}^{2}}\right). \tag{21}$$

Replacing element 9 from Eq. (20) with expression (21) we obtain the so-called modified dispersion equation:

$$-2\left(\frac{\varepsilon_{co}}{\varepsilon_{cl}}J^{-}+H^{-}\right)\left(\frac{\mu_{cl}}{\mu_{co}}J^{+}+H^{+}\right)-2\left(\frac{\varepsilon_{co}}{\varepsilon_{cl}}J^{+}+H^{+}\right)\left(\frac{\mu_{cl}}{\mu_{co}}J^{-}+H^{-}\right)+$$

$$+\underbrace{\frac{4m^{2}}{a^{4}}\left(\frac{\varepsilon_{co}}{\varepsilon_{cl}}\cdot\frac{1}{\chi_{co}^{2}}-\frac{1}{\chi_{cl}^{2}}\right)\left(\frac{\mu_{cl}}{\mu_{co}}\frac{1}{\chi_{cl}^{2}}-\frac{1}{\chi_{cl}^{2}}\right)}_{10}=\underbrace{\left[\frac{2k_{0}n_{cl}\beta m}{a^{2}\chi_{co}^{2}\chi_{cl}^{2}}\left(\frac{n_{co}^{2}}{n_{cl}^{2}}-1\right)\right]^{2}}_{11}. \tag{22}$$

Equation (22) differs very much from the modified dispersion equation for a conventional fiber given in the Marcuse book [14]. As we are looking for cutoff conditions, it is of interest to see what happens with elements 10 and 11 of Eq. (22) when the Hankel function's argument vanishes.

With the help of relations (5)–(6) element 11 can be changed in the following way:

$$\left[\frac{2k_{0}n_{cl}\beta m}{a^{2}\chi_{co}^{2}\chi_{cl}^{2}}\left(\frac{n_{co}^{2}}{n_{cl}^{2}}-1\right)\right]^{2}=\left[\frac{2k_{0}n_{cl}\beta m}{a^{2}\chi_{co}^{2}\chi_{cl}^{2}}\left(\frac{n_{co}^{2}-n_{cl}^{2}}{n_{cl}^{2}}\right)\right]^{2}\times\frac{k_{0}^{2}}{k_{0}^{2}}=$$

$$=\left[\frac{2k_{0}^{2}\beta m\left(n_{co}^{2}-n_{cl}^{2}\right)}{a^{2}\chi_{co}^{2}\chi_{cl}^{2}n_{cl}k_{0}}\right]^{2}=\left[\frac{2\beta m\left(\chi_{co}^{2}+\chi_{cl}^{2}\right)}{a^{2}\chi_{co}^{2}\chi_{cl}^{2}n_{cl}k_{0}}\right]^{2}. \tag{23}$$

We will compose the difference between element 10 and expression (23) as

$$\frac{4m^{2}}{a^{4}}\left(\frac{\varepsilon_{co}}{\varepsilon_{cl}}\cdot\frac{1}{\chi_{co}^{2}}-\frac{1}{\chi_{cl}^{2}}\right)\left(\frac{\mu_{co}}{\mu_{cl}}\frac{1}{\chi_{co}^{2}}-\frac{1}{\chi_{cl}^{2}}\right)-\left[\frac{2\beta m\left(\chi_{co}^{2}+\chi_{cl}^{2}\right)}{a^{2}\chi_{co}^{2}\chi_{cl}^{2}n_{cl}k_{0}}\right]^{2}=0. \tag{24}$$

Making simplification in Eq. (24) we obtain:

$$\frac{4m^{2}}{a^{4}}\left(\frac{\left(\varepsilon_{co}\chi_{cl}^{2}-\varepsilon_{cl}\chi_{co}^{2}\right)\left(\mu_{co}\chi_{cl}^{2}-\mu_{cl}\chi_{co}^{2}\right)}{\varepsilon_{cl}\mu_{cl}\chi_{cl}^{4}\chi_{co}^{4}}\right)-\left[\frac{2\beta m\left(\chi_{co}^{2}+\chi_{cl}^{2}\right)}{a^{2}\chi_{co}^{2}\chi_{cl}^{2}n_{cl}k_{0}}\right]^{2}=$$

$$=\frac{4m^{2}}{a^{4}}\left(\frac{\varepsilon_{co}\mu_{co}\chi_{cl}^{4}-\varepsilon_{co}\chi_{cl}^{2}\mu_{cl}\chi_{co}^{2}-\varepsilon_{cl}\chi_{co}^{2}\mu_{co}\chi_{cl}^{2}+\varepsilon_{cl}\mu_{cl}\chi_{co}^{4}}{n_{cl}^{2}\chi_{cl}^{4}\chi_{co}^{4}}\right)-$$

$$-\frac{4\beta^{2}m^{2}\left(\chi_{co}^{4}+2\chi_{cl}^{2}\chi_{co}^{2}+\chi_{cl}^{4}\right)}{a^{4}n_{cl}^{2}\chi_{cl}^{4}\chi_{co}^{4}k_{0}^{2}}=0. \tag{25}$$

When $\chi_{cl} \to 0$, from relation (8) one can see that $\beta^2 \approx k_0^2 n_{cl}^2$; respectively, $\frac{\beta^2}{k_0^2 n_{cl}^2} = 1$. From Eq. (25) follows the limit for $\chi_{cl} \to 0$:

$$\lim_{\chi_{cl}\to 0}\left(\frac{4m^2 n_{cl}^2 \chi_{co}^4}{\chi_{cl}^4 a^4 n_{cl}^2 \chi_{co}^4} - \frac{4m^2 \chi_{co}^4}{a^4 \chi_{cl}^4 \chi_{co}^4}\right) = \lim_{\chi_{cl}\to 0}\left(\frac{4m^2}{a^4 \chi_{cl}^4} - \frac{4m^2}{a^4 \chi_{cl}^4}\right) = 0. \tag{26}$$

This means that the solution of the modified dispersion equation at cutoff frequencies is independent of elements 10 and 11; we therefore can obtain the simple form:

$$\left(\frac{\varepsilon_{co}}{\varepsilon_{cl}} J^- + H^-\right)\left(\frac{\mu_{co}}{\mu_{cl}} J^+ + H^+\right) + \left(\frac{\varepsilon_{co}}{\varepsilon_{cl}} J^+ + H^+\right)\left(\frac{\mu_{co}}{\mu_{cl}} J^- + H^-\right) = 0. \tag{27}$$

Since the argument of Hankel's functions vanishes in this case, we need to approximate them for small arguments. In terms of introduced abbreviations we will have [14]:

$$\lim_{q\to 0} H^+ = -\frac{2m}{(q)^2} \qquad m = 1,2,3...; \tag{28}$$

$$\lim_{q\to 0} H^- = -\ln\frac{\Gamma q}{2} \qquad m = 1; \tag{29}$$

$$\lim_{q\to 0} H^- = \frac{1}{2(m-1)} \qquad m = 2,3,4..., \tag{30}$$

where $q = \chi_{cl} a$ and $\Gamma = 1.781672$.

The approximations for Bessel's functions of small argument are [9]:

$$J_0(p) = 1 \qquad m = 0; \tag{31}$$

$$J_m(p) = \frac{1}{m!}\left(\frac{p}{2}\right)^m \qquad m = 1,2,3..., \tag{32}$$

where $p = \chi_{co} a$.

Equation (27) can be rewritten with the help of Eq. (11) as

$$\frac{\left(\frac{\varepsilon_{co}}{\varepsilon_{cl}} J_{m-1}(p) + pJ_m(p)H^-\right)\left(\frac{\mu_{co}}{\mu_{cl}} q^2 J_{m+1}(p) - 2mpJ_m(p)\right)}{p^2 q^2 J_m^2(p)} +$$

$$+\frac{\left(\frac{\varepsilon_{co}}{\varepsilon_{cl}} q^2 J_{m+1}(p) - 2mpJ_m(p)\right)\left(\frac{\mu_{co}}{\mu_{cl}} J_{m-1}(p) + pJ_m(p)H^-\right)}{p^2 q^2 J_m^2(p)} = 0. \tag{33}$$

After multiplication by $p^2q^2J_m^2(p)$ and using approximation (30) for small values of q we obtain:

$$pJ_m(p)\left(J_{m-1}(p)\left(\frac{\varepsilon_{co}}{\varepsilon_{cl}}+\frac{\mu_{co}}{\mu_{cl}}\right)+\frac{pJ_m(p)}{(m-1)}\right)=0. \tag{34}$$

This equation admits two solutions when $m>1$:

$$J_m(p)=0, \qquad \text{for } p\neq 0; \tag{35}$$

$$\left(\frac{\varepsilon_{co}}{\varepsilon_{cl}}+\frac{\mu_{co}}{\mu_{cl}}\right)J_{m-1}(p)=\frac{p}{(1-m)}J_m(p). \tag{36}$$

In order to see why $p = 0$ has to be excluded from the solution, we have to use formulas (31, 32) for Bessel functions at small values of p.

When $p\to 0$ and $q\to 0$, for $m > 1$ with the help of Eqs. (28), (30) and (32) we obtain:

$$\left(\frac{\varepsilon_{co}}{\varepsilon_{cl}}\frac{2m}{p^2}+\frac{1}{2(m-1)}\right)\left(\frac{\mu_{co}}{\mu_{cl}}\frac{1}{2(m+1)}-\frac{2m}{q^2}\right)+$$
$$+\left(\frac{\varepsilon_{co}}{\varepsilon_{cl}}\frac{1}{2(m+1)}-\frac{2m}{q^2}\right)\left(\frac{\mu_{co}}{\mu_{cl}}\frac{2m}{p^2}+\frac{1}{2(m-1)}\right)=0. \tag{37}$$

At p and q approaching zero, Eq. (37) reduces to

$$\frac{4m^2\varepsilon_{co}}{p^2q^2\varepsilon_{cl}}+\frac{4m^2\mu_{co}}{p^2q^2\mu_{cl}}=0 \quad\Rightarrow\quad \frac{4m^2\left(\dfrac{\varepsilon_{co}}{\varepsilon_{cl}}+\dfrac{\mu_{co}}{\mu_{cl}}\right)}{p^2q^2}=0. \tag{38}$$

As we can see, when p and q are vanishing the solution $p = 0$ cannot be that of the dispersion equation in the cutoff region.

4. SOLUTION FOR $m = 1$

Now we will find a solution for the cylindrical function index $m = 1$. Here, another way of thinking could be applied, similar to that demonstrated in work [18] by Schelkunoff.

First, we will rewrite dispersion equation (4) in terms of p and q instead of χ_{co} and χ_{cl}:

$$\left[\frac{\mu_{co}}{p}\frac{J'_m(p)}{J_m(p)}-j\frac{\mu_{cl}}{q}\frac{H'^{(1)}_m(jq)}{H^{(1)}_m(jq)}\right]\cdot\left[\frac{\varepsilon_{co}}{p}\frac{J'_m(p)}{J_m(p)}-j\frac{\varepsilon_{cl}}{q}\frac{H'^{(1)}_m(jq)}{H^{(1)}_m(jq)}\right]=$$
$$=\left(\frac{1}{p^2}+\frac{1}{q^2}\right)^2\frac{\beta^2m^2}{k_0^2}. \tag{39}$$

Now we will open the brackets in the left part and modify the right part of Eq. (39) with the help of Eqs. (5, 6):

$$\frac{\varepsilon_{co}\mu_{co}}{p^2}\frac{J_m'^2(p)}{J_m^2(p)}-\frac{\varepsilon_{cl}\mu_{cl}}{q^2}\frac{H_m'^{(1)2}(jq)}{H_m^{(1)2}(jq)}-j\frac{H_m'^{(1)}(jq)J_m'(p)}{H_m'^{(1)}(jq)J_m'(p)}\frac{(\varepsilon_{co}\mu_{cl}+\varepsilon_{cl}\mu_{co})}{pq}=$$

$$=\left(\frac{1}{p^2}+\frac{1}{q^2}\right)\left(\frac{\varepsilon_{co}\mu_{co}}{p^2}+\frac{\varepsilon_{cl}\mu_{cl}}{q^2}\right)m^2. \tag{40}$$

Using the relations from [17]:

$$Z_m'(z)=-\frac{m}{z}Z_m(z)+Z_{m-1}(z)=\frac{m}{z}Z_m(z)-Z_{m+1}(z); \tag{41}$$

$$2\frac{m}{z}Z_m(z)=Z_{m-1}(z)+Z_{m+1}(z); \tag{42}$$

$$Z_m'^{\,2}(z)=\frac{m^2}{z^2}Z_m^2(z)-Z_{m-1}(z)Z_{m+1}(z) \tag{43}$$

we can replace the squared cylindrical function derivatives in Eq. (40) thus obtaining:

$$-\frac{\mu_{co}\varepsilon_{co}J_{m-1}(p)J_{m+1}(p)}{p^2J_m^2(p)}+\overbrace{\frac{\mu_{co}\varepsilon_{co}}{p^4}m^2}^{12}+\frac{\mu_{cl}\varepsilon_{cl}H_{m-1}^{(1)}(jq)H_{m+1}^{(1)}(jq)}{q^2H_m^{(1)2}(jq)}+$$

$$+\overbrace{\frac{\mu_{cl}\varepsilon_{cl}}{q^4}m^2}^{13}-j\frac{J_m'(p)H_m'^{(1)}(jq)}{J_m(p)H_m^{(1)}(jq)}\left(\frac{\varepsilon_{co}\mu_{cl}+\varepsilon_{cl}\mu_{co}}{pq}\right)=$$

$$=\left(\underbrace{\frac{\varepsilon_{co}\mu_{co}}{p^4}}_{14}+\underbrace{\frac{\varepsilon_{cl}\mu_{cl}}{q^4}}_{15}+\frac{\varepsilon_{co}\mu_{co}}{p^2q^2}+\frac{\varepsilon_{cl}\mu_{cl}}{p^2q^2}\right)m^2 \tag{44}$$

The next step is to transfer elements 12 and 13 to the right side of Eq. (44), because they are equal to elements 14 and 15; thus we can reduce them as

$$-\frac{\mu_{co}\varepsilon_{co}J_{m-1}(p)J_{m+1}(p)}{p^2J_m^2(p)}+\frac{\mu_{cl}\varepsilon_{cl}H_{m-1}^{(1)}(jq)H_{m+1}^{(1)}(jq)}{q^2H_m^{(1)2}(jq)}-$$

$$-j\frac{J_m'(p)H_m'^{(1)}(jq)}{J_m(p)H_m^{(1)}(jq)}\left(\frac{\varepsilon_{co}\mu_{cl}+\varepsilon_{cl}\mu_{co}}{pq}\right)=\left(\frac{\varepsilon_{co}\mu_{co}+\varepsilon_{cl}\mu_{cl}}{p^2q^2}\right)m^2. \tag{45}$$

From Hankel's function theory it is simple to derive the following relations:

$$\frac{H_m'^{(1)}(jq)}{qH_m^{(1)}(jq)}=j\frac{1}{q^2}+\frac{H_{m-1}^{(1)}(jq)}{qH_m^{(1)}(jq)}; \tag{46}$$

$$\frac{H_{m+1}^{(1)}(jq)}{q^2}=-\frac{H_{m-1}^{(1)}(jq)}{q^2}-j\frac{2m}{q^3}H_m^{(1)}(jq). \tag{47}$$

Substituting them in Eq. (45) we obtain

$$-\frac{\mu_{co}\varepsilon_{co}J_{m-1}(p)J_{m+1}(p)}{p^2J_m^2(p)}-\frac{\mu_{cl}\varepsilon_{cl}H_{m-1}^{(1)}(jq)}{H_m^{(1)^2}(jq)}\left(\frac{H_{m-1}^{(1)}(jq)}{q^2}+j\frac{2m}{q^3}H_m^{(1)}(jq)\right)-$$

$$-j\frac{J_m'(p)}{J_m(p)}\left(\frac{\varepsilon_{co}\mu_{cl}+\varepsilon_{cl}\mu_{co}}{p}\right)\left(j\frac{1}{q^2}+\frac{H_{m-1}^{(1)}(jq)}{qH_m^{(1)}(jq)}\right)=\left(\frac{\varepsilon_{co}\mu_{co}+\varepsilon_{cl}\mu_{cl}}{p^2q^2}\right)m^2 \tag{48}$$

As we are interested in the case of $m=1$, Eq. (48) takes the form:

$$\frac{\mu_{co}\varepsilon_{co}J_0(p)J_2(p)}{p^2J_1^2(p)}q^2+\frac{\mu_{cl}\varepsilon_{cl}H_0^{(1)}(jq)}{H_1^{(1)^2}(jq)}\left(H_0^{(1)}(jq)+j\frac{2}{q}H_1^{(1)}(jq)\right)+$$

$$+j\frac{J_1'(p)}{J_1(p)}\left(\frac{\varepsilon_{co}\mu_{cl}+\varepsilon_{cl}\mu_{co}}{p}\right)\left(j+q\frac{H_0^{(1)}(jq)}{H_1^{(1)}(jq)}\right)=-\left(\frac{\varepsilon_{co}\mu_{co}+\varepsilon_{cl}\mu_{cl}}{p^2}\right). \tag{49}$$

At $q\to 0$ this equation reduces to the following:

$$-\frac{J_1'(p)}{J_1(p)}\left(\frac{\varepsilon_{co}\mu_{cl}+\varepsilon_{cl}\mu_{co}}{p}\right)+\underbrace{\varepsilon_{cl}\mu_{cl}\frac{H_0^{(1)}(jq)}{H_1^{(1)^2}(jq)}\left(H_0^{(1)}(jq)+j\frac{2H_1^{(1)}(jq)}{q}\right)}_{16}=$$

$$=-\left(\frac{\varepsilon_{co}\mu_{ci}+\varepsilon_{cl}\mu_{cl}}{p^2}\right). \tag{50}$$

For Hankel's functions the following approximation relations [14] exist:

$$\lim_{q\to 0}H_0^{(1)}(jq)=\frac{2j}{\pi}\ln\left(\frac{\Gamma q}{2}\right)\qquad m=0; \tag{51}$$

$$\lim_{q\to 0}H_m^{(1)}(jq)=-\frac{j(m-1)!}{\pi}\left(\frac{2}{jq}\right)^m. \tag{52}$$

We can approximate element 16 of the left side of Eq. (50) for small values of q as

$$\varepsilon_{cl}\mu_{cl}H_0^{(1)}(jq)\left(\frac{H_0^{(1)}(jq)}{H_1^{(1)^2}(jq)}+j\frac{2}{qH_1^{(1)}(jq)}\right)=$$

$$= \varepsilon_{cl}\mu_{cl}H_0^{(1)}(jq)\left(\frac{H_0^{(1)}(jq)}{\left(-\frac{j}{\pi}\frac{2}{jq}\right)^2}+\frac{j2}{q\left(-\frac{j}{\pi}\frac{2}{jq}\right)}\right)=$$

$$= \varepsilon_{cl}\mu_{cl}H_0^{(1)}(jq)\left(\frac{\pi^2 q^2 H_0^{(1)}(jq)}{4}-j\pi\right) \tag{53}$$

for which approximation formula (52) is used.

Finally, the 16^{th} element of Eq. (50) takes the form:

$$\varepsilon_{cl}\mu_{cl}H_0^{(1)}(jq)\left(\frac{\pi^2 q^2 \frac{2j}{\pi}\ln\frac{\Gamma q}{2}}{4}-j\pi\right)=-j\pi\varepsilon_{cl}\mu_{cl}H_0^{(1)}(jq) \tag{54}$$

because q reaches zero faster than the logarithm grows. Equation (50) after multiplication by p^2 is written as

$$\underbrace{-(\varepsilon_{co}\mu_{cl}+\varepsilon_{cl}\mu_{co})\frac{J_1'(p)}{J_1(p)}p}_{17} =$$

$$= jp^2\pi\varepsilon_{cl}\mu_{cl}H_0^{(1)}(jq)-\varepsilon_{co}\mu_{co}-\varepsilon_{cl}\mu_{cl}. \tag{55}$$

The right part of Eq. (55) for small q values tends to infinity. The left part will reach infinity only when the denominator is zero. This means that the solution will be

$$J_1(p)=0. \tag{56}$$

With the help of Eq. (41) we can regroup elements in Eq. (55) in the following way:

$$-(\varepsilon_{co}\mu_{cl}+\varepsilon_{cl}\mu_{co})\frac{J_0(p)}{J_1(p)p}+\frac{\varepsilon_{co}\mu_{cl}+\varepsilon_{cl}\mu_{co}}{p^2}+\frac{\varepsilon_{co}\mu_{c0}+\varepsilon_{cl}\mu_{cl}}{p^2}=$$

$$= j\pi\varepsilon_{cl}\mu_{cl}H_0^{(1)}(jq). \tag{57}$$

It is only the left part of Eq. (57) that is dependent on p – that is, this left part is a function $F(p)$. In Fig. 2 there are plotted $F(p)$ and $J_1(p)$ as functions of p. From Fig. 2 one can see that when p reaches the value of the root of Eq. (56) (the dashed line crosses the p-axis) the value of $F(p)$ (a solid line) tends either to plus or minus infinity. This fact is a confirmation that Eq. (56) is the solution of Eq. (55), because the right part of Eq. (57) tends to plus infinity when q vanishes.

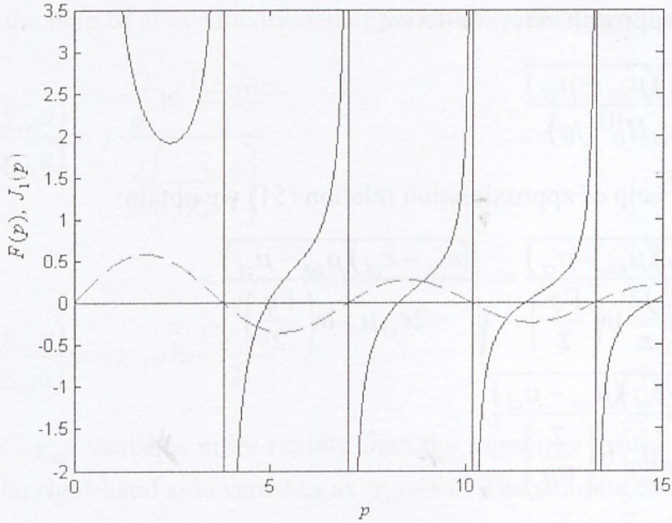

Fig. 2. The plots of $F(p)$ (solid line) and $J_1(p)$ (dashed line). $|\varepsilon_{co}| = 2.25$; $|\mu_{co}| = 1.11$; $\varepsilon_{cl} = 1.25$; $\mu_{cl} = 1$

The smallest value of p is zero. In Fig. 2 the function $F(p)$ tends to infinity when p is vanishingly small. Let us prove analytically that this fact holds true. We have already known that when $m > 1$, $p = 0$ cannot be a solution. In this case for element 17 of Eq. (55) the following is valid:

$$-(\varepsilon_{co}\mu_{cl} + \varepsilon_{cl}\mu_{co})\frac{J_1'(p)}{J_1(p)}p = -(\varepsilon_{co}\mu_{cl} + \varepsilon_{cl}\mu_{co})\frac{1}{J_1(p)}p\left(-\frac{J_1(p)}{p} + J_0(p)\right) =$$

$$= -(\varepsilon_{co}\mu_{cl} + \varepsilon_{cl}\mu_{co})\left(-1 + \frac{pJ_0(p)}{J_1(p)}\right) =$$

$$= (\varepsilon_{co}\mu_{cl} + \varepsilon_{cl}\mu_{co}) - (\varepsilon_{co}\mu_{cl} + \varepsilon_{cl}\mu_{co})p^2 J^- . \tag{58}$$

Here Eqs.(11) and (41) were used.

When $p \to 0$ we obtain:

$$(\varepsilon_{co}\mu_{cl} + \varepsilon_{cl}\mu_{co}) - (\varepsilon_{co}\mu_{cl} + \varepsilon_{cl}\mu_{co})p^2 J^- =$$

$$= (\varepsilon_{co}\mu_{cl} + \varepsilon_{cl}\mu_{co}) - (\varepsilon_{co}\mu_{cl} + \varepsilon_{cl}\mu_{co})p^2 \frac{2}{p^2} = -\varepsilon_{co}\mu_{cl} - \varepsilon_{cl}\mu_{co} \tag{59}$$

due to the approximation [14]:

$$\lim_{p \to 0} J^- = \frac{2}{p^2}, \qquad m = 1. \tag{60}$$

Using the result of expression (59) we reduce Eq. (55) to

$$(\varepsilon_{co} - \varepsilon_{cl})(\mu_{co} - \mu_{cl}) = jp^2 \pi \varepsilon_{cl}\mu_{cl} H_0^{(1)}(jq), \tag{61}$$

and as q and p approach zero, we derive:

$$p=\sqrt{\frac{(\varepsilon_{co}-\varepsilon_{cl})(\mu_{co}-\mu_{cl})}{j\pi\varepsilon_{cl}\mu_{cl}H_0^{(1)}(jq)}}\,. \tag{62}$$

With the help of approximation relation (51) we obtain:

$$p=\sqrt{\frac{(\varepsilon_{co}-\varepsilon_{cl})(\mu_{co}-\mu_{cl})}{j\pi\varepsilon_{cl}\mu_{cl}\frac{j2}{\pi}\ln\left(\frac{\Gamma q}{2}\right)}}=\sqrt{\frac{(\varepsilon_{co}-\varepsilon_{cl})(\mu_{co}-\mu_{cl})}{-2\varepsilon_{cl}\mu_{cl}\ln\left(\frac{\Gamma q}{2}\right)}}=$$

$$=\sqrt{\frac{(\varepsilon_{co}-\varepsilon_{cl})(\mu_{co}-\mu_{cl})}{2\varepsilon_{cl}\mu_{cl}\ln\left(\frac{2}{\Gamma q}\right)}}\,. \tag{63}$$

Equation (63) is the evidence that when $m=1$, $p=0$ is the solution of the eigenvalue equation, because Hankel's function values tend to infinity when q vanishes. As p is related to the cutoff frequency by the equation [14]:

$$f_c=\frac{c}{2\pi}\frac{p_c}{a\sqrt{n_{co}^2-n_{cl}^2}}\,. \tag{64}$$

this means that for the optical fiber with a negative index of refraction there exists a mode with the theoretically zeroth cutoff frequency as it is for conventional fibers.

5. SOLUTION FOR $m = 0$

Finally, we have to investigate the case of $m=0$. The dispersion equation (4) will be divided into two parts as

$$\left[-a\frac{\varepsilon_{co}\chi_{cl}^2}{\varepsilon_{cl}\chi_{co}}\frac{J_1(\chi_{co}a)}{J_0(\chi_{co}a)}+ja\chi_{cl}\frac{H_1^{(1)}(j\chi_{cl}a)}{H_0^{(1)}(j\chi_{cl}a)}\right]=0;$$

$$\left[-a\frac{\mu_{co}\chi_{cl}^2}{\mu_{cl}\chi_{co}}\frac{J_1(\chi_{co}a)}{J_0(\chi_{co}a)}+ja\chi_{cl}\frac{H_1^{(1)}(j\chi_{cl}a)}{H_0^{(1)}(j\chi_{cl}a)}\right]=0\,. \tag{65}$$

Here, the relation [17]

$$Z_0'(z)=-Z_1(z) \tag{66}$$

was used. It suffices to rewrite one of Eqs. (65) (since they differ only by factors $\varepsilon_{co}/\varepsilon_{cl}$ and μ_{co}/μ_{cl}):

$$\frac{\varepsilon_{cl}\chi_{co}}{\varepsilon_{co}\chi_{cl}}\frac{J_0(\chi_{co}a)}{J_1(\chi_{co}a)}=\frac{H_0^{(1)}(j\chi_{cl}a)}{jH_1^{(1)}(j\chi_{cl}a)}\,. \tag{67}$$

With the help of above mentioned approximation relations we derive:

$$\frac{\varepsilon_{cl}\chi_{co}}{\varepsilon_{co}\chi_{cl}}\frac{J_0(\chi_{co}a)}{J_1(\chi_{co}a)} = j\frac{\frac{j2}{\pi}\ln\frac{\Gamma\chi_{cl}a}{2}}{\frac{j}{\pi}\left(\frac{2}{j\chi_{cl}a}\right)} \tag{68}$$

and

$$\frac{\varepsilon_{cl}\chi_{co}}{\varepsilon_{co}\chi_{cl}}\frac{J_0(\chi_{co}a)}{J_1(\chi_{co}a)} = -\chi_{cl}a\ln\frac{\Gamma\chi_{cl}a}{2}. \tag{69}$$

Since $a\chi_{cl}$ vanishes more rapidly than the logarithm grows to infinity, the product of the right-hand side vanishes as $\chi_{cl} \to 0$. The solution of this equation is thus

$$J_0(\chi_{co}a) = 0. \tag{70}$$

It is easy to see that vanishingly small values of χ_{co} do not give a sulution of Eq. (65) since:

$$\lim_{p\to 0}\frac{\varepsilon_{cl}p}{\varepsilon_{co}}\frac{J_0(p)}{J_1(p)} = \lim_{p\to 0}\frac{\varepsilon_{cl}p}{2\varepsilon_{co}p} = \frac{\varepsilon_{cl}}{2\varepsilon_{co}} \neq 0 \tag{71}$$

and this means that $\chi_{co} = 0$ (or $p = 0$) is not the solution of the dispersion equation when $m = 0$.

6. RESULTS AND DISCUSSION

As the result of calculations, the cutoff behaviour for all the modes of a cladded optical fiber with an LHM core can be described to a sufficient degree. Below, the cutoff solutions of the eigenvalue equation for the guided modes of an optical fiber with a negative refraction index are summarized. In Table 1 all formulas for every mode in the LHM core fiber are presented with comparison to those of a conventional fiber. Also, for the conventional fiber the following abbreviations are used for the modes: transverse electrical *TE*, transverse magnetic *TM*, and hybrid modes *HE* and *EH*. In this table the parameter $p_{m\xi}$ is the ξ-th root of the corresponding equation containing the cutoff condition, with ξ being an integer. The formulas for the conventional fiber are taken from [13, 14, 15].

From the solution of the eigenvalue equation we can see that for the considered type of fiber the modes are the same as for conventional fibers. From the comparison of the cutoff conditions for the LHM core fiber and the conventional fiber given in Table 1 it is easy to see that cutoff conditions are different only for hybrid modes $HE_{m\xi}$ $(m > 1)$. This means that these modes have different cutoff frequencies, while other modes have identical cutoff frequencies. In Table 2 the roots of Eq. (36) and of the equation describing the cutoff condition of the conventional fiber are given. The core and cladding parameters are: $|\varepsilon_{co}| = 2.25$; $|\mu_{co}| = 1.11$; $\varepsilon_{cl} = 1.25$; $\mu_{cl} = 1$. The roots are calculated for the hybrid modes $HE_{2\xi}$ $(m = 2)$.

Table 1

Cutoff conditions for a conventional fiber and a fiber with a negative index of refraction

Number of the equation	m	ξ	Mode	Fiber with LHM core	Conventional fiber
(70)	0	>0	$TE_{0\xi}$ $TM_{0\xi}$	$J_0(p_{0\xi}) = 0$	$J_0(p_{0\xi}) = 0$
(63)	1	1	HE_{11}	$p_{11} = 0$	$p_{11} = 0$
(56)	1	>1	$HE_{1\xi}$	$J_1(p_{1\xi}) = 0$	$J_1(p_{1\xi}) = 0$
(56)	1	>0	$EH_{1\xi}$	$J_1(p_{1\xi}) = 0 \quad (p_{11} \neq 0)$	$J_1(p_{1\xi}) = 0 \quad (p_{11} \neq 0)$
(36)	>1	>0	$HE_{m\xi}$	$\left(\frac{\varepsilon_{co}}{\varepsilon_{cl}} + \frac{\mu_{co}}{\mu_{cl}}\right) J_{m-1}(p_{m\xi}) = = -\frac{p_{m\xi}}{(m-1)} J_m(p_{m\xi})$ $p_{m\xi} \neq 0$	$\left(\frac{\varepsilon_{co}}{\varepsilon_{cl}} + \frac{\mu_{co}}{\mu_{cl}}\right) J_{m-1}(p_{m\xi}) = = \frac{p_{m\xi}}{(m-1)} J_m(p_{m\xi})$ $p_{m\xi} \neq 0$
(35)	>1	>0	$EH_{m\xi}$	$J_m(p_{m\xi}) = 0 (p_{m\xi} \neq 0)$	$J_m(p_{m\xi}) = 0 (p_{m\xi} \neq 0)$

Table 2

The roots of cutoff equations for hybrid modes $HE_{2\xi}$ of the conventional fiber and the LHM core fiber

$HE_{2\xi}$	ξ=1	ξ=2	ξ=3
Conventional fiber	2.69	5.65	8.74
LHM core fiber	4.65	8.11	11.37

From Table 2 one can see that the roots of the equation describing the cutoff conditions for $HE_{m\xi}$ modes ($m > 1$) in the LHM core fiber are larger than those for the classical fiber, so the cutoff frequencies for these modes turn out to be larger than usual. This means that these modes appear later, when we increase the frequency of electromagnetic wave as compared with the conventional case.

7. CONCLUSION

In the present paper, the methods are given for finding analytical solution of the dispersion equation for different values of the cylinder function index. Here it is proved that when $m = 0$ and m>1 there is no mode with the zero cutoff frequency. This inference follows from Eqs.(38) and (71). At the same time, Eq. (63) shows that only when $m = 1$ (as in conventional fibers) there exists the mode HE_{11} with theoretical $f_c = 0$. This important conclusion makes it possible to predict that in an LHM core fiber a frequency region exists where the wave via the fiber can be transmitted with one mode, i.e. a single mode regime can be defined.

The paper gives an introduction to the guided mode theory in fibers with a negative index of refraction. It helps one to better understand the analytical solution of the dispersion equation right at cutoff frequencies, both for fibers of isotropic left-handed materials and for conventional fibers. In order to obtain more results, we need to find a numerical solution of the dispersion equation.

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DISPERSIJAS VIENĀDOJUMA ANALĪTISKS ATRISINĀJUMS UN ROBEŽFREKVENCES OPTISKAI ŠĶIEDRAI AR NEGATĪVU LAUŠANAS KOEFICIENTU

V. Butenko

Kopsavilkums

Darbā atrisināts dispersijas vienādojums pakāpes optiskajai šķiedrai gadījumā, kad šķiedras serde ir no materiāla ar negatīvu laušanas koeficientu $n_{co} < 0$, bet apvalks ir no parasta dielektriskā materiāla ar pozitīvu laušanas koeficientu $n_{cl} > 0$. Serdes negatīvajam laušanas koeficientam atbilst negatīvas dielektriskās un magnētiskās caurlaidības vērtības. Izpētīta negatīvās dielektriskās $\varepsilon_{co} < 0$ un magnētiskās $\mu_{co} < 0$ caurlaidības ietekme uz šķērselektriskām *TE*, šķērsmagnētiskām *TM* un hibrīdām (*EH* un *HE*) gaismas viļņu modām šķiedrā. Dispersijas vienādojuma atrisinājums ir salīdzināts ar tā atrisinājumu parastās pakāpes šķiedras gadījumā, kad gan serdei, gan apvalkam ir pozitīvs laušanas koeficients. Pierādīts, ka aplūkotajā šķiedrā ar negatīvu laušanas koeficientu eksistē hibrīdmodas HE_{11} ar robežfrekvenci, kas ir vienāda ar nulli.

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ENERĢĒTIKAS FIZIKĀLĀS UN TEHNISKĀS PROBLĒMAS

CIETVIELU FIZIKA